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9:37 AM

Unit-1.

Def. Exact differential equation: A differential Equation of type $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be exact if there exist, a function $f(x,y)$ such that differential eq. can be written as $d[f(x,y)] = 0$ and in that case solution to the differential eq. is $f(x,y) = C$

$$M(x,y)\,dx = -N(x,y)\,dy \qquad \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)}, \quad N(x,y) \neq 0$$

$$\boxed{\frac{dy}{dx} = \phi(x,y)}$$

$x\,dy + y\,dx = 0$

$$y\,dx + x\,dy = 0$$

$$M\,dx + N\,dy = 0$$

$$f(x,y) = xy$$

$$d(xy) = x\,dy + y\,dx$$

$$d(xy) = 0 \rightarrow d f(x,y) = 0 \qquad \int d(x) = x$$

EXACT

$xy = C \rightarrow$ SOLUTION

$$M\,dx + N\,dy = 0 \Rightarrow d[f(x,y)] = 0$$

$$\int d\, f(x,y) = C$$

$$\Rightarrow f(x,y) = C$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x) = x^{-7/2}$$

$$f(x) = x^n$$
$$f'(x) = nx^{n-1}$$

$$-\tfrac{7}{2}x^{-9/2}$$

$M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be exact if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$\underbrace{y}_{M}\,dx + \underbrace{x}_{N}\,dy = 0 \rightarrow$ EXACT

$M = y$ & $N = x$

$$\frac{\partial M}{\partial y} = 1 \quad \& \quad \frac{\partial N}{\partial x} = 1 \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

22/01/21 # If $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact then general solution of the differential equation is

$$\int_{y=\text{constant}} M\,dx + \int (\text{Terms of } N \text{ not containing } x)\,dy = C$$

Q→ Check whether the following differential equations are exact and obtain the general solution

(i) $(1+e^x)\,dx + y\,dy = 0$ (ii) $\left[3x^2y + y/x\right]dx + \left[x^3 + \log x\right]dy = 0$

(iii) $(2x + e^y)\,dx + x e^y\,dy = 0$

(iii) $(2x+e^{y})dx + xe^{y}\,dy = 0$

Sol. (i) $(1+e^{x})dx + y\,dy = 0$ Compare it $M(x,y)dx + N(x,y)dy = 0$

$\Rightarrow M = 1+e^{x}$ & $N = y$

Now $\frac{\partial M}{\partial y} = 0$ & $\frac{\partial N}{\partial x} = 0 \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ Given differential equation is exact.

Now solution of given differential eq.

$$\int_{y=\text{constant}} M\,dx + \int [\text{Terms of } N \text{ not containing } x]\,dy = C$$

$$\Rightarrow \int_{y=\text{const}} (1+e^{x})\,dx + \int y\,dy = C \Rightarrow \left[x + e^{x} + \frac{y^2}{2}\right] = C$$

(ii) $\left[3x^2y + \frac{y}{x}\right]dx + \left[x^3 + \log x\right]dy = 0$

Compare it $M\,dx + N\,dy = 0 \Rightarrow M = 3x^2y + \frac{y}{x}$ & $N = x^3 + \log x$

Now $\frac{\partial M}{\partial y} = 3x^2\cdot 1 + \frac{1}{x}\cdot 1$ & $\frac{\partial N}{\partial x} = 3x^2 + \frac{1}{x}$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ Given differential eq. is exact.

Now solution of given diff eq. is $\int_{y=\text{const}} M\,dx + \int [\text{Terms of } N \text{ not containing } x]\,dy = c$

$$\Rightarrow \int_{y=\text{const}} \left(3x^2y + \frac{y}{x}\right)dx + 0 = C \Rightarrow \cancel{3}y\frac{x^3}{\cancel{3}} + y\log|x| = c$$

$$\Rightarrow x^3y + y\log x = c$$

(iv) $x\,dy - y\,dx = e^{y}(x^2+y^2)\,dy$

$\Rightarrow -y\,dx + \left[x - e^{y}x^2 - e^{y}y^2\right]dy = 0$

Compare it with $M\,dx + N\,dy = 0 \Rightarrow M = -y$ & $N = x - x^2e^{y} - y^2e^{y}$

Now $\frac{\partial M}{\partial y} = -1$ & $\frac{\partial N}{\partial x} = 1 - e^{y}(2x) - 0$

Clearly $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ Given differential eq. is not exact

Integrating factor: Let $M(x,y)dx + N(x,y)dy = 0$ is not exact, then a term $\mu(x,y)$ which when multiplied with the given differential equation, makes it exact, is called an integrating factor.

$\frac{dy}{dx} + Py = Q$

I.F. $= e^{\int P\,dx}$

Rules to find Integrating factor:

(1) If both $M(x,y)$ & $N(x,y)$ in $M\,dx + N\,dy = 0$ are homogeneous functions of degree n, then

integrating factor $= I.F. = \frac{1}{Mx+Ny}$, $Mx+Ny \neq 0$

Q→ Solve $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$ — (i)

Sol.: Compare it with $Mdx + Ndy = 0$

$M = x^2y - 2xy^2$ & $N = -x^3 + 3x^2y$

Now $\frac{\partial M}{\partial y} = x^2 - 2x(2y)$ & $\frac{\partial N}{\partial x} = -3x^2 + 3y(2x)$

$= x^2 - 4xy$ $\qquad = -3x^2 + 6xy$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ Eq (i) is not exact

Here $M(x,y) = x^2y - 2xy^2$ & $N(x,y) = -x^3 + 3x^2y$

Now $M(\lambda x, \lambda y) = (\lambda x)^2(\lambda y) - 2(\lambda x)(\lambda y)^2$

$= \lambda^2 x^2 \lambda y - 2\lambda x \lambda^2 y^2$

$= \lambda^3 [x^2y - 2xy^2]$

$= \lambda^3 M(x,y)$

$\therefore M(\lambda x, \lambda y) = \lambda^3 M(x,y)$

$f(\lambda x, \lambda y) = \lambda^n f(x,y)$

$\therefore f(x,y)$ is homogenous with degree $= n$

$\therefore M(x,y)$ is homogeneous function with degree $= 3$

Again $N(\lambda x, \lambda y) = -\lambda^3 x^3 + 3\lambda^3 x^2 y = \lambda^3[-x^3 + 3x^2y]$

$= \lambda^3 N(x,y)$

$\therefore N(\lambda x, \lambda y) = \lambda^3 N(x,y)$

$\therefore N(x,y)$ is a homogeneous function with degree $= 3$

$\therefore$ Integrating factor $= I.F. = \frac{1}{Mx+Ny} = \frac{1}{\cancel{x^3y} - 2x^2y^2 - \cancel{x^3y} + 3x^2y^2}$

$\therefore I.F. = \frac{1}{x^2y^2}$

Multiplying given diff eq (i) with $\frac{1}{x^2y^2}$

$(x^2y - 2xy^2)dx - [x^3 - 3x^2y]dy = 0$

$\therefore \left(\frac{1}{y} - \frac{2}{x}\right)dx - \left[\frac{x}{y^2} - \frac{3}{y}\right]dy = 0$ — (ii)

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Compare it $M'dx + N'dy = 0$

$\therefore M' = \frac{1}{y} - \frac{2}{x}$ & $N' = -\frac{x}{y^2} + \frac{3}{y}$

$\therefore \frac{\partial M'}{\partial y} = -\frac{1}{y^2}$ & $\frac{\partial N'}{\partial x} = -\frac{1}{y^2} \Rightarrow \frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$

$\therefore$ (ii) is exact, therefore solution to eq. (ii) is

$\int_{y=\text{Const}} M'dx + \int [\text{Terms of } N' \text{ without } x]\, dy = C$

$$\Rightarrow \int_{y=\text{Const}} \left(\frac{1}{y} - \frac{2}{x}\right) dx + \int \frac{3}{y}\, dy = C$$

$$\Rightarrow \frac{1}{y}x - 2\log x + 3\log y = C \Rightarrow \frac{x}{y} - 2\log x + 3\log y = C$$

Rule 2: If $Mdx+Ndy=0$ is of type $\Rightarrow f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$

then I.F. $= \frac{1}{Mx-Ny}$, $Mx-Ny \neq 0$

Solve

Q $\rightarrow$ $(x^3y^2+x)dy + (x^2y^3-y)dx = 0$ — (i)

Sol $\rightarrow$ Here $M = x^2y^3-y$ & $N = x^3y^2+x$

$\frac{\partial M}{\partial y} = 3x^2y^2-1$ & $\frac{\partial N}{\partial x} = 3x^2y^2+1$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ (i) is not exact

Now (i) $\Rightarrow (x^2y^2+1)x\,dy + (x^2y^2-1)y\,dx = 0$

$\therefore$ I.F $= \frac{1}{Mx-Ny} = \frac{1}{(x^3y^3-xy)-(x^3y^3+xy)}$

$= \frac{1}{\cancel{x^3y^3}-xy-\cancel{x^3y^3}-xy} = -\frac{1}{2xy}$

Multiply (i) with $-\frac{1}{2xy}$

$\left[-\frac{x^2y}{2} - \frac{1}{2y}\right]dy + \left[-\frac{xy^2}{2} + \frac{1}{2x}\right]dx = 0$ — (ii)

Compare it with $M'dx + N'dy = 0$

Here $M' = -\frac{xy^2}{2} + \frac{1}{2x}$ & $N' = -\frac{x^2y}{2} - \frac{1}{2y}$

$\therefore \frac{\partial M'}{\partial y} = -\frac{x}{2}(2y) = -xy$

& $\frac{\partial N'}{\partial x} = -\frac{y}{2}(2x) = -xy$

$\therefore \frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x} \Rightarrow$ (ii) is exact.

$\therefore$ Solution to eq. (ii) is

$$\int_{y=\text{Const}} M'dx + \int [\text{Terms of } N' \text{ without } x]\,dy = C$$

$$\Rightarrow \int_{y=\text{Const}} \left[-\frac{xy^2}{2} + \frac{1}{2x}\right] dx + \int -\frac{1}{2y}\, dy = C$$

$$\Rightarrow -\frac{y^2}{2}\cdot\frac{x^2}{2} + \frac{1}{2}\log x - \frac{1}{2}\log y = C$$

$M(\lambda x, \lambda y)$

$= \lambda^2x^2\lambda^3y^3 - \lambda y$

$= \lambda^5x^2y^3 - \lambda y$

$= \lambda[\lambda^4x^2y^3 - y]$

$f(\lambda x, \lambda y) = \lambda^n f(x,y)$

$f(x,y) = x^2+3y^2-2x^1y^1$

$k = (x^2y^2+1)$

$[xy = t]$

$k = [t^2+1] = \phi(t)$

$= \phi(xy)$

$p = (x^2y^2-1)$

$p = t^2-1 = h(t)$

$= h(xy)$

$(x^3y^2+x)dy + (x^2y^3-y)dx = 0$

Rule 3 : If $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N} = f(x)$ [Function of x only], then

$$I.F = e^{\int f(x)\,dx}$$

Rule 4 : If $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{M} = g(y)$ [Function of y only], then

$$I.F = e^{-\int g(y)\,dy}$$

Solve

Q→ $(y^4+2y)dx + (xy^3+2y^4-4x)dy = 0$ —(i)

Sol → $M = y^4+2y$ & $N = xy^3+2y^4-4x$

$\dfrac{\partial M}{\partial y} = 4y^3+2$ & $\dfrac{\partial N}{\partial x} = y^3-4$

$\Rightarrow \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x} \Rightarrow$ (i) is not exact.

Now $\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x} = (4y^3+2)-(y^3-4) = 3y^3+6$

$$\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{M} = \frac{3y^3+6}{y[y^3+2]} = \frac{3\cancel{[y^3+2]}}{y\cancel{[y^3+2]}} = \frac{3}{y}$$

$$\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N} = \frac{3y^3+6}{xy^3+2y^4-4x}$$

$\therefore$ $I.F = e^{-\int \frac{3}{y}dy} = e^{-3\log y} = e^{\log(y^{-3})}$

$e^{\log a} = a$

$= y^{-3} = \dfrac{1}{y^3}$

Multiplying (i) $\dfrac{1}{y^3}$

$(y^4+2y)dx + (xy^3+2y^4-4x)dy = 0$

$\Rightarrow \left[y+\dfrac{2}{y^2}\right]dx + \left[x+2y-\dfrac{4x}{y^3}\right]dy = 0$ —(ii)

Compare with $M'dx + N'dy = 0 \Rightarrow M' = y+\dfrac{2}{y^2}$ & $N' = x+2y-\dfrac{4x}{y^3}$

Now $\dfrac{\partial M'}{\partial y} = 1-\dfrac{4}{y^3}$ & $\dfrac{\partial N'}{\partial x} = 1-\dfrac{4}{y^3}$

$\therefore \dfrac{\partial M'}{\partial y} = \dfrac{\partial N'}{\partial x} \Rightarrow$ (ii) is exact

$\therefore$ Solution to (ii) is $\displaystyle\int_{y=\text{Const}} M'dx + \int[\text{Terms of } N' \text{ without } x]dy = c$

$$\Rightarrow \int_{y=\text{Const}} \left(y+\frac{2}{y^2}\right)dx + \int 2y\,dy = c$$

$$\Rightarrow \left(y+\frac{2}{y^2}\right)x + y^2 = c$$

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Q. Solve $(4xy + 3y^2 - x)dx + x(x+2y)dy = 0$ — (i)

Sol. Compare it with $Mdx + Ndy = 0$.

$\Rightarrow M = 4xy + 3y^2 - x$ & $N = x^2 + 2xy$

$\frac{\partial M}{\partial y} = 4x + 6y$ & $\frac{\partial N}{\partial x} = 2x + 2y \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ (i) is not exact

$f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$

Now $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4x + 6y - 2x - 2y$

$= 2x + 4y = 2[x + 2y]$

Now $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \dfrac{2\cancel{[x+2y]}}{x\cancel{[x+2y]}} = \frac{2}{x}$

Now I.F. $= e^{\int \frac{2}{x} dx} = e^{2\log x} = e^{\log x^2}$ | $e^{\log a} = a$

$= x^2$

Now multiply (i) with x^2

$\therefore$ (i) $\Rightarrow [4x^3y + 3x^2y^2 - x^3]dx + [x^4 + 2x^3y]dy = 0$ — (ii)

Compare it $M'dx + N'dy = 0$

$M' = 4x^3y + 3x^2y^2 - x^3$ & $N' = x^4 + 2x^3y$

Now $\frac{\partial M'}{\partial y} = 4x^3 + 6x^2y$ & $\frac{\partial N'}{\partial x} = 4x^3 + 6x^2y$

$\therefore \frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x} \Rightarrow$ (ii) is exact

$\therefore$ Solution to (ii) is $\int_{y=\text{Const}} M'dx + \int [\text{Terms of } N \text{ without } x]dy = c$

$\Rightarrow \int_{y=\text{Const}} [4x^3y + 3x^2y^2 - x^3]dx + 0 = C$

$\Rightarrow 4y\frac{x^4}{4} + 3y^2 \cdot \frac{x^3}{3} - \frac{x^4}{4} = c$

$\Rightarrow x^4y + x^3y^2 - \frac{x^4}{4} = c$

Rule II: If $Mdx + Ndy = 0$ is of type

$[x^a y^b [my\,dx + nx\,dy] + x^{a'} y^{b'} [m'y\,dx + n'x\,dy] = 0]$

then I.F. $= x^h y^k$

where $\frac{a+h+1}{m} = \frac{b+k+1}{n}$ & $\frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}$

Q. Solve $(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0$ — (i)

Sol. Compare with $Mdx + Ndy = 0$

$\Rightarrow M = y^2 + 2x^2y$ & $N = 2x^3 - xy$

$\frac{\partial M}{\partial y} = 2y + 2x^2$ & $\frac{\partial N}{\partial x} = 6x^2 - y \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ (i) is not exact

(i) $\Rightarrow y^2dx + 2x^2y\,dx + 2x^3\,dy - xy\,dy = 0$ $\quad f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$

$x^0y[y\,dx - x\,dy] + x^2[2y\,dx + 2x\,dy] = 0$ $\quad [y + 2x^2]\,y\,dx$

Compare it with

$x^ay^b[my\,dx + nx\,dy] + x^{a'}y^{b'}[m'y\,dx + n'x\,dy] = 0$

$xy = t$ $\quad f(t)$

$\left[y + 2\frac{t^2}{y^2}\right] \Rightarrow f(t, y)$

$\left[\frac{t}{x} + 2x^2\right] \to f(x, t)$

$a = 0, b = 1, m = 1, n = -1$

$a' = 2, b' = 0, m' = 2, n' = 2$

$\therefore$ I.F. $= x^h y^k$

where $\frac{a+h+1}{m} = \frac{b+k+1}{n}$

$\Rightarrow \frac{0+h+1}{1} = \frac{1+k+1}{-1}$

$\Rightarrow h+1 = -k-2$

$\Rightarrow h+k = -3$ — (ii)

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2y + 2x^2 - 6x^2 + y$
$= 3y - 4x^2$

$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{3y - 4x^2}{2x^3 - xy} \to f(x)$

$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{3y - 4x^2}{y^2 + 2x^2y} \to f(y)$

again $\frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}$

$\Rightarrow \frac{2+h+1}{\cancel{2}} = \frac{0+k+1}{\cancel{2}} \Rightarrow h - k = -2$ — (iii)

(ii) + (iii) $\Rightarrow 2h = -5 \Rightarrow \boxed{h = -5/2}$ & $k = -3 - h$
$= -3 + 5/2 = -1/2$
$\Rightarrow \boxed{k = -1/2}$

$\therefore$ I.F. $= x^hy^k = x^{-5/2}y^{-1/2} = \frac{1}{x^{5/2}y^{1/2}}$

Multiply (i) with $\frac{1}{x^{5/2}y^{1/2}}$ $\quad (y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0$

$\therefore$ (i) $\Rightarrow \left[\frac{y^{3/2}}{x^{5/2}} + \frac{2y^{1/2}}{x^{1/2}}\right]dx + \left[\frac{2x^{1/2}}{y^{1/2}} - \frac{y^{1/2}}{x^{3/2}}\right]dy = 0$ — (ii)

Compare it $M'dx + N'dy = 0$

$M' = \frac{y^{3/2}}{x^{5/2}} + \frac{2y^{1/2}}{x^{1/2}}$ & $N' = \frac{2x^{1/2}}{y^{1/2}} - \frac{y^{1/2}}{x^{3/2}}$

$\frac{\partial M'}{\partial y} = \frac{3}{2}\frac{y^{1/2}}{\ldots} \cdot \cancel{2}\cdot\frac{1}{2}y^{-1/2} \ldots = \frac{3}{2}\frac{y^{1/2}}{\ldots} + \frac{1}{\ldots}$

$\frac{\partial y}{} - \quad 2\frac{}{x^{5/2}} + \quad \frac{}{x^{1/2}} \quad 2\,x^{5/2} \quad x^{1/2}y^{1/2}$

and $\frac{\partial N'}{\partial x} = \cancel{2}\frac{1}{\cancel{2}}\frac{x^{-1/2}}{y^{1/2}} - \left(-\frac{3}{2}\right)x^{-5/2}\cdot y^{1/2} = \underbrace{\frac{1}{x^{1/2}y^{1/2}}} + \underbrace{\frac{3}{2}\cdot\frac{y^{1/2}}{x^{5/2}}}$

$\therefore \frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x} \Rightarrow$ (ii) is exact

$\therefore$ Solution to (ii) is $\int_{y=\text{const}} M'dx + \int[\text{Terms of } N' \text{ without } x]dy = C$

$$\Rightarrow \int_{y=\text{Const}}\left[\frac{y^{3/2}}{x^{5/2}} + \frac{2y^{1/2}}{x^{1/2}}\right]dx = C$$

$$\Rightarrow y^{3/2}\cdot\frac{x^{-3/2}}{-3/2} + 2y^{1/2}\frac{x^{1/2}}{1/2} = C$$

$$\Rightarrow -\frac{2}{3}\frac{y^{3/2}}{x^{3/2}} + 4x^{1/2}y^{1/2} = C$$

Rule-VI: # (Finding I.F by inspection)

$\underbrace{x\,dy + y\,dx} = \underbrace{d(xy)}$

$\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$ # $\frac{x\,dy - y\,dx}{y^2} = -d\left(\frac{x}{y}\right)$

$\frac{x\,dy - y\,dx}{xy} = d\left[\log\left(\frac{y}{x}\right)\right]$ # $\frac{x\,dy - y\,dx}{x^2+y^2} = d\left[\tan^{-1}\left(y/x\right)\right]$

$\frac{x\,dy - y\,dx}{x^2-y^2} = d\left[\frac{1}{2}\log\left(\frac{x+y}{x-y}\right)\right]$

$d(xy) = x\,dy + y\,dx$ # $\frac{x\,dy - y\,dx}{x^2} = d(y/x)$

$\frac{x\,dy - y\,dx}{y^2} = -d(x/y) = -\left[\frac{y\,dx - x\,dy}{y^2}\right] = \frac{x\,dy - y\,dx}{y^2}$

$\frac{x\,dy - y\,dx}{xy} = d[\log(y/x)] = d[\log y - \log x] = \frac{1}{y}dy - \frac{1}{x}dx$

$$= \frac{x\,dy - y\,dx}{xy}$$

Q.→ $x\,dy - y\,dx = e^{y}(x^2+y^2)\,dy$

Sol.: $\frac{x\,dy - y\,dx}{x^2+y^2} = e^{y}\,dy \Rightarrow d[\tan^{-1}(y/x)] = d(e^{y})$

Integrating

$\tan^{-1}(y/x) = e^{y} + C$ $\quad \int d[\tan^{-1}(y/x)]$

Q. $x\,dy - y\,dx + y^2 dx = 0$

Sol. $x\,dy - y\,dx = -y^2 dx$

$$\Rightarrow \frac{x\,dy - y\,dx}{y^2} = -dx$$

$$\Rightarrow \frac{y\,dx - x\,dy}{y^2} = dx$$

$$\Rightarrow d\left(\frac{x}{y}\right) = dx$$

Integrating

$$\Rightarrow \frac{x}{y} = x + c$$

$$= \frac{1}{1+(y/x)^2}\, d\left(y/x\right)$$

$$= \frac{x^2}{x^2+y^2}\left[\frac{x\,dy - y\,dx}{x^2}\right]$$

$$= \frac{x\,dy - y\,dx}{x^2+y^2}$$

Equations of first order & higher degree:

eg. $y\left(\frac{dy}{dx}\right)^2 - 2xy = 0 \rightarrow$ order $= 1$, degree $= 2$

To solve diff. equations of first order & higher degree, we write $\frac{dy}{dx} = p$, then equation takes a form as

$$f(x, y, p) = 0 \quad \text{--- (1)}$$

$yp^2 - 2xy = 0$

$f(x, y, p) = 0$

Case 1: Equations solvable for p:

Factorise (1), in linear factors as

$$[p - f_1(x,y)][p - f_2(x,y)] \text{ ---} = 0$$

$$\Rightarrow p = f_1(x,y),\quad p = f_2(x,y) \text{ ---}$$

$p = f_1(x,y)$

$\frac{dy}{dx} = f_1(x,y)$

All these are diff. eqn. of first order & first degree

Let solution to these equations is

$$F_1(x, y, c) = 0,\quad F_2(x, y, c_2) = 0$$

Then general solutions of (1) will be

$$F_1(x,y,c) \cdot F_2(x,y,c) \text{ --} = 0$$

Q. Solve $y\left(\frac{dy}{dx}\right)^2 - (x-y)\frac{dy}{dx} - x = 0$

Sol. This is equation of 1st order & 2nd degree (Higher degree)

Take $\frac{dy}{dx} = p$

$$\Rightarrow yp^2 - xp + yp - x = 0$$

$\Rightarrow \quad p(yp-x)+(yp-x)=0 \Rightarrow (p+1)(yp-x)=0$

$\therefore \quad p+1=0$

$\Rightarrow \frac{dy}{dx}+1=0$

$\Rightarrow \frac{dy}{dx}=-1$

$\Rightarrow dy=-dx$

Integrate

$y=-x+C_1$

$\Rightarrow y+x-C_1=0$

$yp-x=0$

$\Rightarrow y\frac{dy}{dx}-x=0$

$\Rightarrow y\frac{dy}{dx}=x$

$\Rightarrow y\,dy=x\,dx \longrightarrow d\left(\frac{y^2}{2}\right)=d\left(\frac{x^2}{2}\right)$

Integrating

$\frac{y^2}{2}=\frac{x^2}{2}+C_2$

$\Rightarrow \frac{y^2}{2}-\frac{x^2}{2}-C_2=0$

$\therefore$ General solution to starting eq. is

$$(y+x-C_1)\left(\frac{y^2}{2}-\frac{x^2}{2}-C_2\right)=0$$

Q → $p^3+2xp^2-y^2p^2-2xy^2p=0, \quad p=\frac{dy}{dx}$

$\left(\frac{dy}{dx}\right)^3+2x\left(\frac{dy}{dx}\right)^2-y^2\left(\frac{dy}{dx}\right)^2-2xy^2\left(\frac{dy}{dx}\right)=0$

Sol. → $p\left[p^2+2xp-y^2p-2xy^2\right]=0$

$\Rightarrow p\left[p(p+2x)-y^2(p+2x)\right]=0$

$\Rightarrow p(p+2x)(p-y^2)=0$

either

$p=0$

$\frac{dy}{dx}=0$

$\Rightarrow dy=0$

$\Rightarrow y=C_1$

$\Rightarrow y-C_1=0$

or $p+2x=0$

$\Rightarrow \frac{dy}{dx}=-2x$

$\Rightarrow dy=-2x\,dx$

Integrate

$y=-\cancel{2}\frac{x^2}{\cancel{2}}+C_2$

$y+x^2-C_2=0$

or $p-y^2=0$

$\Rightarrow \frac{dy}{dx}=y^2$

$\Rightarrow \frac{dy}{y^2}=dx$

Integrate

$-\frac{1}{y}=x+C_3$

$\Rightarrow x+\frac{1}{y}+C_3=0$

$\therefore$ General solution to given eq. is

$$(y-C_1)(y+x^2-C_2)\left(x+\frac{1}{y}+C_3\right)=0$$

H.W.

Q → (i) $p(p+y)=x(x+y)$

(ii) $y=x\left[p+\sqrt{1+p^2}\right]$ (iii) $xy\left(\frac{dy}{dx}\right)^2-(x^2+y^2)\frac{dy}{dx}+xy=0$

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Case 2 : Equations solvable for y :

For a diff equation of first order & higher degree, we take $\frac{dy}{dx}=p$

then eq. takes a form $f(x,y,p)=0$ ← (1)

If from above equation we can get $y=\phi(x,p)$

If from above equation we can get $y = \phi(x, p)$
diff w.r.t. x

$$\frac{dy}{dx} = g\left(x, p, \frac{dp}{dx}\right)$$

$$\Rightarrow p = g\left(x, p, \frac{dp}{dx}\right)$$

solve this differential in p & x, let
$\psi(p, x) = c$ is the solution to above eq. (ii)

$\therefore$ Eliminate p from (i) & (ii), to get general solution of eq. (i)

\# In case, if we will not be able to eliminate p from eq. (i) & (ii), we simply (i) & (ii) together give us the general solution.

Q → Solve $\quad y - 2px = \tan^{-1}[p^2 x]$ — (i)

Sol → $\Rightarrow y = 2px + \tan^{-1}(p^2 x)$

diff w.r.t. x

$$\Rightarrow \frac{dy}{dx} = 2\left[p \cdot 1 + x\frac{dp}{dx}\right] + \frac{1}{1+(p^2x)^2}\left[p^2 \cdot 1 + x \cdot 2p\frac{dp}{dx}\right]$$

$$\Rightarrow p = 2p + 2x\frac{dp}{dx} + \frac{p^2}{1+p^4x^2} + \frac{2px\frac{dp}{dx}}{1+p^4x^2}$$

$$\Rightarrow -p - \frac{p^2}{1+p^4x^2} = \left[2x + \frac{2px}{1+p^4x^2}\right]\frac{dp}{dx}$$

$$\Rightarrow -p\left[1 + \frac{p}{1+p^4x^2}\right] = 2x\left[1 + \frac{p}{1+p^4x^2}\right]\frac{dp}{dx}$$

$$\Rightarrow 2x\left[1 + \frac{p}{1+p^4x^2}\right]\frac{dp}{dx} + p\left[1 + \frac{p}{1+p^4x^2}\right] = 0$$

$$\Rightarrow \left(1 + \frac{p}{1+p^4x^2}\right)\left(2x\frac{dp}{dx} + p\right) = 0$$

$$\Rightarrow 2x\frac{dp}{dx} + p = 0 \quad \Rightarrow \quad 2x\frac{dp}{dx} = -p$$

$$\Rightarrow \frac{dp}{p} = -\frac{dx}{2x}$$

Integrating

$$\Rightarrow \log p = -\frac{1}{2}\log x + \log c$$

$$\Rightarrow \log p = \log(x^{-1/2}) + \log c = \log c x^{-1/2}$$

$y = 0$

$\log a + \log b = \log ab$

$$\Rightarrow p = c x^{-1/2} \quad \text{--- (ii)}$$

Put $p = c x^{-1/2}$ in (i) $\Rightarrow$ $y - 2 c x^{-1/2} \cdot x = \tan^{-1}[x\, c^2 x^{-1}]$

$$\Rightarrow y - 2c\sqrt{x} = \tan^{-1}[c^2]$$

is the general solution

for diff. eq. given in (i)

Q→ Solve $y = x + a \tan^{-1} p$ --- (i)

Sol: diff w.r.t x

$$\Rightarrow \frac{dy}{dx} = 1 + a \frac{1}{1+p^2} \frac{dp}{dx}$$

$$\Rightarrow p = 1 + \frac{a}{1+p^2}\frac{dp}{dx} \Rightarrow (p-1) = \frac{a}{1+p^2}\frac{dp}{dx}$$

$$\Rightarrow \frac{a\,dp}{(1+p^2)(p-1)} = dx$$

Integrating

$$\Rightarrow a \int \frac{dp}{(p^2+1)(p-1)} = \int dx + C$$

$$\Rightarrow a I = x + C \quad \text{--- (ii)}, \text{ where } I = \int \frac{dp}{(p^2+1)(p-1)}$$

Now, $\frac{1}{(p^2+1)(p-1)} = \frac{A}{p-1} + \frac{Bp+D}{p^2+1}$

$$\Rightarrow 1 = A(p^2+1) + (Bp+D)(p-1)$$

$$\Rightarrow 1 = Ap^2 + A + Bp^2 - Bp + Dp - D$$

Compare Coeffi. of p^2, $A + B = 0$ --- (iii)

———— p, $-B + D = 0$ --- (iv)

Compare Constants, $A - D = 1$ --- (v)

(iii) + (iv) $\Rightarrow$ $A + D = 0 \Rightarrow A = -D$ Put in (v)

$\therefore$ (v) $\Rightarrow$ $-D - D = 1 \Rightarrow \boxed{D = -1/2}$ and $\boxed{A = 1/2}$

$$\boxed{B = -1/2}$$

$$\therefore \frac{1}{(p-1)(p^2+1)} = \frac{1}{2(p-1)} + \frac{-\frac{1}{2}p - \frac{1}{2}}{p^2+1}$$

$$\Rightarrow \int \frac{1}{(p-1)(p^2+1)} dp = \frac{1}{2}\int \frac{1}{p-1} dp - \frac{1}{2}\int \frac{p+1}{p^2+1} dp$$

$$\Rightarrow I = \frac{1}{2}\log(p-1) - \frac{1}{2}\int \frac{p}{p^2+1} dp - \frac{1}{2}\int \frac{1}{p^2+1} dp$$

$$= \frac{1}{2}\log(p-1) - \frac{1}{2}\times\frac{1}{2}\int \frac{2p}{p^2+1} dp - \frac{1}{2}\tan^{-1} p$$

$I = \frac{1}{2}\log(p-1) - \frac{1}{4}\log|p^2+1| - \frac{1}{2}\tan^{-1}p$

Put in (ii)

$\left[\int \frac{f'(x)}{f(x)}dx = \log|f(x)| \right]$

$\therefore$ (ii) $a\left[\frac{1}{2}\log(p-1) - \frac{1}{4}\log(p^2+1) - \frac{1}{2}\tan^{-1}p\right] = x+c$ — (vi)

From (i) $y = x + a\tan^{-1}p \Rightarrow \frac{y-x}{a} = \tan^{-1}p \Rightarrow p = \tan\left(\frac{y-x}{a}\right)$

Put in (vi)

$$a\left[\frac{1}{2}\log\left[\tan\left(\frac{y-x}{a}\right)-1\right] - \frac{1}{4}\log\left[\tan^2\left(\frac{y-x}{a}\right)+1\right] - \frac{1}{2}\left(\frac{y-x}{a}\right)\right] = x+c$$

Rule-3 : Equations solvable for x :

Let equation with first order & higher degree be comes

$f(x, y, p) = 0$ — (i)

If from above equation, we can get $x = \phi(y, p)$ — (i)

diff w.r.t y

$\Rightarrow \frac{dx}{dy} = g\left(y, p, \frac{dp}{dy}\right)$

$\Rightarrow \frac{1}{p} = g\left(y, p, \frac{dp}{dy}\right)$

Solve this diff. eq & let solution is

$\psi(p, y, c) = 0$ — (ii)

Eleminate p from (i) & (ii) to get the general solution

H.W. Q→ Solve $y = px + x^4 p^2$

Q→ Solve $x^2\left(\frac{dy}{dx}\right)^4 + 2x\frac{dy}{dx} - y = 0$

30/01/21 Q→ Solve $y = 2px + y^2p^3$ — (i)

Sol.→ $x = \frac{y - y^2p^3}{2p}$

diff w.r.t y

$$\Rightarrow \frac{dx}{dy} = \frac{2p\left[1 - y^2 3p^2\frac{dp}{dy} - p^3 2y\right] - (y - y^2p^3)\cdot 2\frac{dp}{dy}}{4p^2}$$

$$\Rightarrow \frac{1}{p} = \frac{2p - 6p^3y^2\frac{dp}{dy} - 4p^4y - 2y\frac{dp}{dy} + 2y^2p^3\frac{dp}{dy}}{4p^2}$$

$\Rightarrow 4p =$ ''

$\Rightarrow 2p + 4p^4y = -4p^3y^2\frac{dp}{dy} - 2y\frac{dp}{dy}$

$$\Rightarrow \quad 2p[1+2p^3y] = -2y[1+2p^3y]\frac{dp}{dy}$$

$$\Rightarrow \quad 2p\underbrace{[1+2p^3y]} + 2y\underbrace{[1+2p^3y]}\frac{dp}{dy} = 0$$

$$\Rightarrow \quad 2(1+2p^3y)\left(p + y\frac{dp}{dy}\right) = 0 \rightarrow [\text{Neglecting } (1+2p^3y)=0$$

$$\Rightarrow \quad p + y\frac{dp}{dy} = 0 \quad \Rightarrow \quad p = -y\frac{dp}{dy} \Rightarrow \frac{dy}{y} = \frac{-dp}{p}$$

Integrate

$$\Rightarrow \log p = -\log y + \log c$$

$$\Rightarrow \log p = \log\left(\frac{c}{y}\right) \Rightarrow \boxed{p = \frac{c}{y}} \quad \text{—(ii)}$$

Eliminating p from (i) & (ii), we get

$$y = 2\left(\frac{c}{y}\right)x + \left(\frac{c}{y}\right)^2 y^2 = 2\frac{cx}{y} + \frac{c^3}{y}$$

$\Rightarrow y^2 = 2cx + c^3$ is the general solution for (i).

Clairaut's Equation: An differential equation of type $y = px + f(p)$ is called clairaut's equation and general solution to the eq. is $y = cx + f(c)$, where C is an arbitrary constants.

Q→ Solve $xp^2 - yp + a = 0$

Sol. → $xp^2 + a = yp \Rightarrow \underbrace{y = px + \frac{a}{p}}$

The clairauts form, so solution is

$$y = cx + \frac{a}{c}$$

Q→ $p = \log(px - y)$

Sol. → $px - y = e^p \Rightarrow y = px - e^p$, this clairaut's eq.

so general solution is

$$y = cx - e^c$$

H.W. Q→ Solve $y = px + \sqrt{a^2p^2 + b^2}$

Q→ Solve $\sin px \cos y = \cos px \sin y + p$

Sol. → $\sin(px)\cos(y) - \cos(px)\sin(y) = p$

$$\Rightarrow \quad \sin[px - y] = p \quad \Rightarrow \quad px - y = \sin^{-1}p$$

$\Rightarrow y = px - \sin^{-1}p$, It is clairauts form

$\therefore$ General solution is $y = cx - \sin^{-1}c$

Q. $y = x\left[p + \sqrt{1+p^2}\right]$

Sol. $y = px + x\sqrt{1+p^2}$

$\Rightarrow \quad y - px = x\sqrt{1+p^2}$

squaring

$\Rightarrow y^2 + p^2x^2 - 2pyx = x^2\left[1+p^2\right]$

$\Rightarrow y^2 + \cancel{p^2x^2} - 2pyx = x^2 + \cancel{p^2x^2}$

$\Rightarrow \quad y^2 - x^2 - 2pyx = 0 \quad \Rightarrow y^2 - x^2 - 2xy\frac{dy}{dx} = 0$

$\Rightarrow \quad (y^2 - x^2) = 2xy\frac{dy}{dx} \longrightarrow \boxed{\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}}$

$y = vx$

Idea of 10+2

$\Rightarrow (y^2 - x^2)\,dx - 2xy\,dy = 0$ — (i)

Comparing with $Mdx + Ndy = 0$

$M = y^2 - x^2$ & $N = -2xy$

$\frac{\partial M}{\partial y} = 2y$ & $\frac{\partial N}{\partial x} = -2y \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ (1) is not exact

$$I.F. = \frac{1}{Mx+Ny} = \frac{1}{y^2x - x^3 - 2xy^2} = \frac{1}{-xy^2 - x^3}$$

multiply (i) with $-\frac{1}{xy^2+x^3}$

$$\Rightarrow \quad \frac{\left[x^2 - y^2\right]}{xy^2 + x^3}dx + \frac{2xy}{xy^2+x^3}dy = 0 \quad \text{— (ii)}$$

Compare with $M'dx + N'dy = 0$

$$M' = \frac{x^2 - y^2}{xy^2 + x^3} \quad \& \quad N' = \frac{2xy}{xy^2+x^3}$$

$$\frac{\partial M'}{\partial y} = \frac{(xy^2 + x^3)\left[-2y\right] - (x^2 - y^2)\left[2xy\right]}{(xy^2 + x^3)^2}$$

$$= \frac{-2\cancel{xy^3} - 2x^3y - 2x^3y + 2\cancel{xy^3}}{(xy^2 + x^3)^2} = \frac{-4x^3y}{(xy^2 + x^3)^2}$$

$$\text{and } \frac{\partial N'}{\partial x} = \frac{(xy^2 + x^3)\cdot 2y - 2xy\cdot\left[y^2 + 3x^2\right]}{(xy^2+x^3)^2}$$

$$= \frac{2\cancel{xy^3} + 2x^3y - 2\cancel{xy^3} - 6x^3y}{(xy^2 + x^3)^2} = \frac{-4x^3y}{(xy^2+x^3)^2}$$

$\Rightarrow \frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x} \Rightarrow$ (ii) is exact

$\therefore$ Solution is $\displaystyle\int_{y=\text{Const}} M'\,dx + \int [\underbrace{\text{Terms of } N' \text{ without } x}]\,dy = C$

$$\Rightarrow \int_{y=\text{Const}} \left[\frac{x^2-y^2}{xy^2+x^3}\right] dx = C$$

$$\Rightarrow \frac{1}{3}\int_{y=\text{Const}} \frac{3x^2-3y^2}{\underbrace{x^3+xy^2}} dx = C \quad \Rightarrow \frac{1}{3}\int_{y=\text{Const}} \frac{3x^2+y^2-4y^2}{\underbrace{x^3+xy^2}} dx = C$$

$$\Rightarrow \frac{1}{3}\int_{y=\text{Const}} \frac{(3x^2+y^2)}{\underbrace{(x^3+xy^2)}} dx - \frac{4y^2}{3}\int_{y=\text{Const}} \frac{1}{x^3+xy^2} dx = C$$

$$\Rightarrow \frac{1}{3}\log|x^3+xy^2| - \frac{4y^2}{3}\int_{y=\text{Const}} \frac{1}{x(x^2+y^2)} dx = C \quad \text{—(i)}$$

$$\int \frac{f'(x)}{f(x)}dx = \log|f(x)|$$

Now $\displaystyle \underbrace{\frac{1}{x(x^2+y^2)}} = \underbrace{\frac{A}{x} + \frac{Bx+D}{x^2+y^2}}$ [$\rightarrow$ find A, B & D at your own]

$$\text{(i)} \Rightarrow \frac{1}{3}\log|x^3+xy^2| - \frac{4y^2}{3}\int_{y=\text{Const}} \left[\frac{A}{x} + \frac{Bx+D}{x^2+y^2}\right] dx = C$$

$$\Rightarrow \frac{1}{3}\log|x^3+xy^2| - \frac{4y^2}{3}\left[A\log|x| + \frac{B}{2}\int_{y=\text{Const}} \frac{2x}{\underbrace{x^2+y^2}}dx + D\int_{y=\text{Const}} \frac{1}{x^2+y^2}dx\right] = C$$

$$\Rightarrow \frac{1}{3}\log|x^3+xy^2| - \frac{4y^2}{3}\left[A\log x + B\log|x^2+y^2| + \frac{D}{y}\tan^{-1}\left(\frac{x}{y}\right)\right] = C$$